Set-Based State Estimation: A Polytopic Approach

Carlos E. Valero*, Mario E. Villanueva**, Boris Houska**, Radoslav Paulen*

Faculty of Chemical and Food Technology
Slovak University of Technology in Bratislava

School of Information Science and Technology
ShanghaiTech University

Motivation

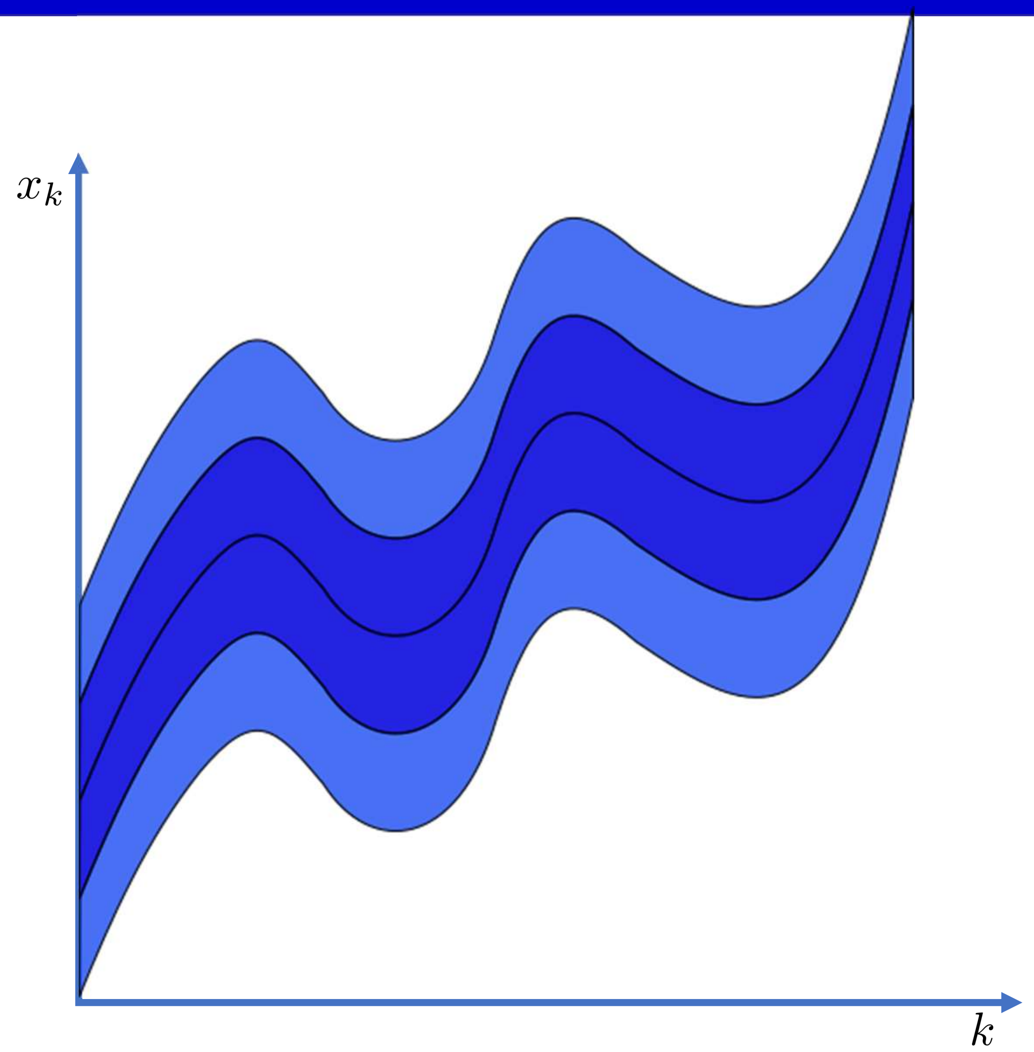
The system

$$x_{k+1} = f(x_k, w_k)$$
$$x_k \in X_k$$
$$w_k \in \mathbb{W}$$

$$y_k = Cx_k + v_k$$
$$v_k \in \mathbb{V}$$

EKF, Luenberger Observer, MHE, SSE, etc.

Set-based State Estimation (SSE)



Set-based State Estimation (SSE)

Construction of the output set

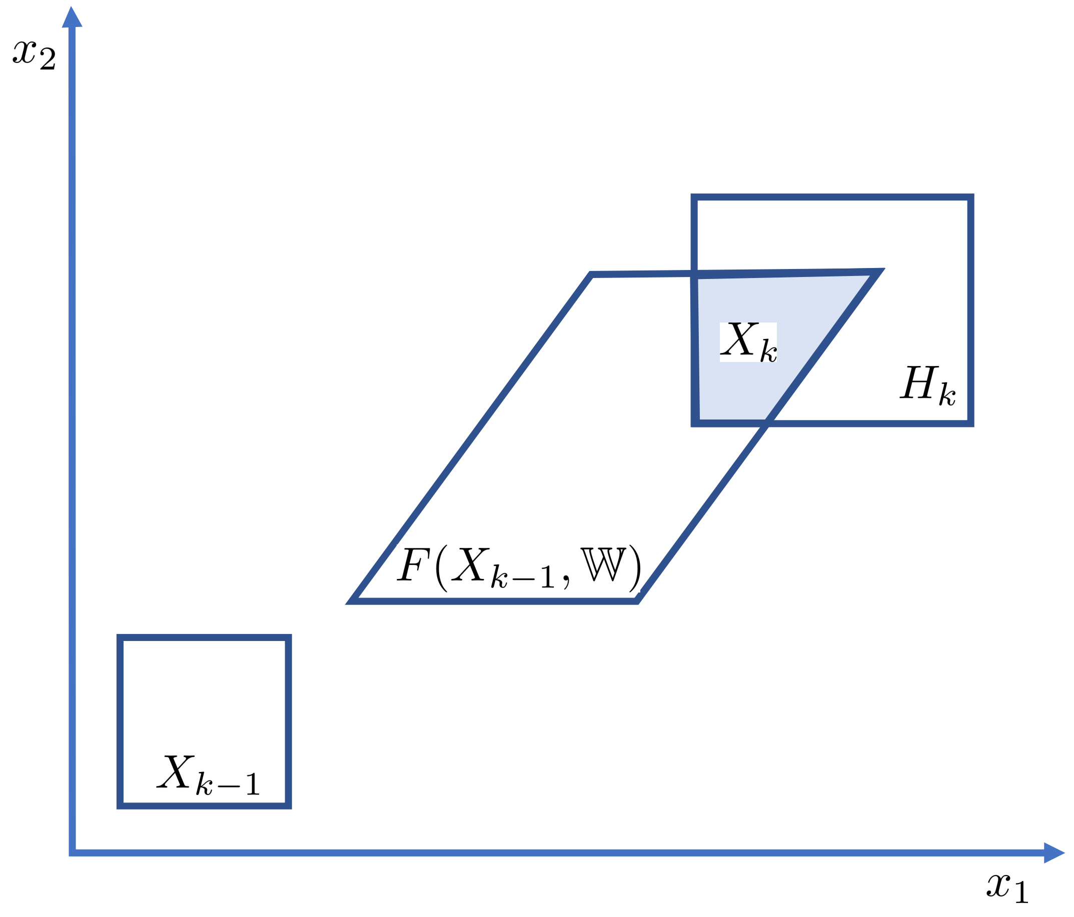
$$H_k = \{\ x \in \mathbb{R}^n \mid Cx - y_k \in \mathbb{V}\ \}$$

Dynamic propagation

$$F(X_{k-1}, \mathbb{W}) = \left\{ f(x_{k-1}, w) \middle| \begin{array}{l} x_{k-1} \in X_{k-1} \\ w \in \mathbb{W} \end{array} \right\}$$

Update process

$$X_k = F(X_{k-1}, \mathbb{W}) \cap H_k$$



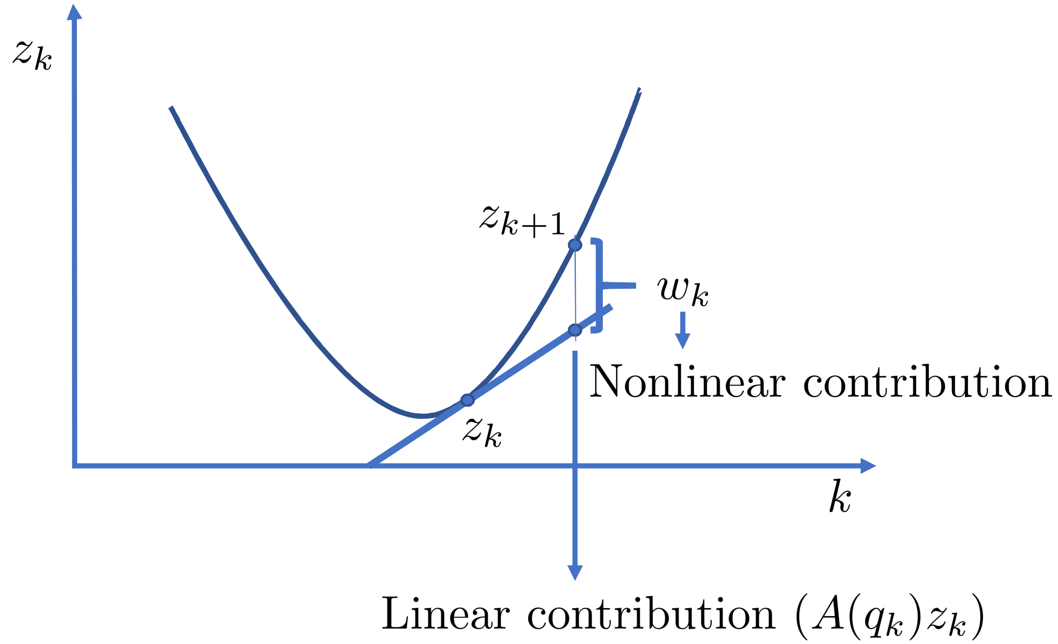
Linear and nonlinear dependencies

$$f(\underbrace{q+z}_{x}, w) = f(q,0) + A(q)z + B(q)w + \eta(q,z,w)$$

$$A = \frac{\partial f}{\partial x}(x,0) \;\text{ and }\; B = \frac{\partial f}{\partial w}(x,0)$$

$$w_k = B(q)w + \eta(q,z,w)$$

$$q_{k+1} = f(q_k, 0)$$
$$z_{k+1} = A(q_k)z_k + \omega_k$$



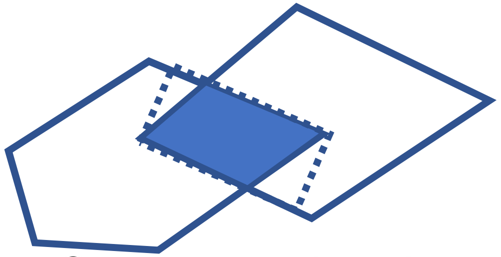
Set-based state estimation for uncertain systems

We can add nonlinearities and uncertainties in a unique set

$$\Omega_k(q_k, Z_k) \supseteq \left\{ \underbrace{B(q_k)w_k}_{\text{Uncertainties}} + \underbrace{\eta_f(q_k, z_k, w_k)}_{\text{Nonlinearities}} \;\middle|\; \begin{array}{l} z_k \in Z_k \\ w_k \in \mathbb{W} \end{array} \right\}$$

The computation of the set where the state lies is given by

$$Z_k = \underbrace{\overbrace{[A(q_{k-1})Z_{k-1} \oplus \Omega_{k-1}(q_{k-1}, Z_{k-1})]}^{\text{Propagation}} \cap [H_k \ominus \{q_k\}]}_{\text{Update}}$$

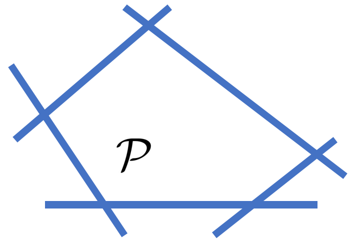


Over-approximation

Polytopes

Half-space representation $\mathcal{H}$-Rep

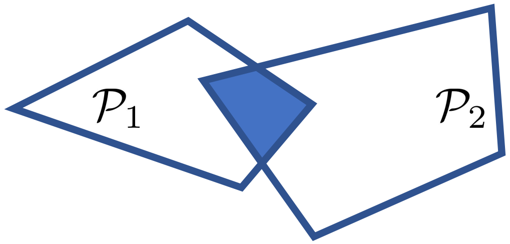
$$\mathcal{P}(G,h) = \{x | Gx \leq h\}$$



Basic operations

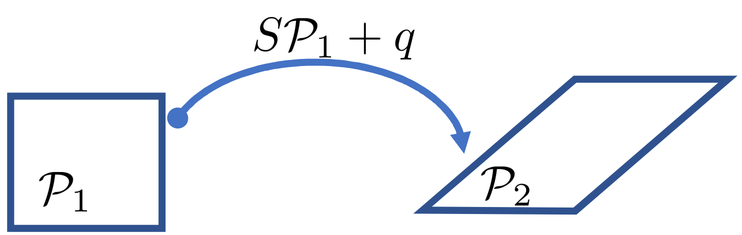
Intersection $\mathcal{P}_1(G_1,h_1)\bigcap\mathcal{P}_2(G_2,h_2)$

$$\mathcal{P}_\cap = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} x \leq \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$



Linear transformation $\mathcal{P}_2(G_2,h_2) = S\mathcal{P}_1(G_1,h_1) + q$

$$\mathcal{P}_2(G_2,h_2) = \mathcal{P}_1(G_1 S^{-1}, h_1 + G_1 S^{-1} q)$$



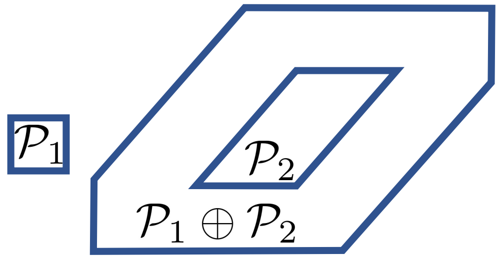
Polytopes

Minkowski sum $\mathcal{P}_1 \oplus \mathcal{P}_2$

$$\mathcal{P}_1(G_1, h_1) \oplus \mathcal{P}_2(G_2, h_2) \subseteq \mathcal{P}(MG_1, Mh_1 + Nh_2)$$

s.t. $MG_1 = NG_2$

Necessary Condition

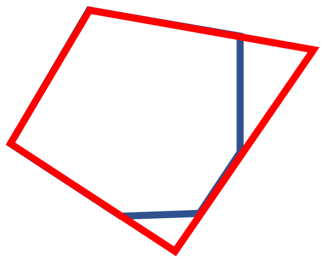


Facet reduction

$$\mathcal{P}_1(G_1, h_1) \subseteq \mathcal{P}_2(G_2, h_2)$$

$$\mathcal{P}_2(\Lambda G_1, \Lambda h_1)$$

Necessary Condition



SSE for Nonlinear Systems using Polytopes

Assumptions

There is a given polytope with uncertainties and nonlinearities.

$$\Omega \subseteq \mathcal{P}_w(G_w h_w)$$

The noise in the measurements is within a known polytope.

$$\mathbb{V} \subseteq \mathcal{P}_v(G_v, h_v)$$

The initial state of the system is contained in a polytope.

$$X_0 \subseteq \mathcal{P}_0(G_0, h_0)$$

Propagation step

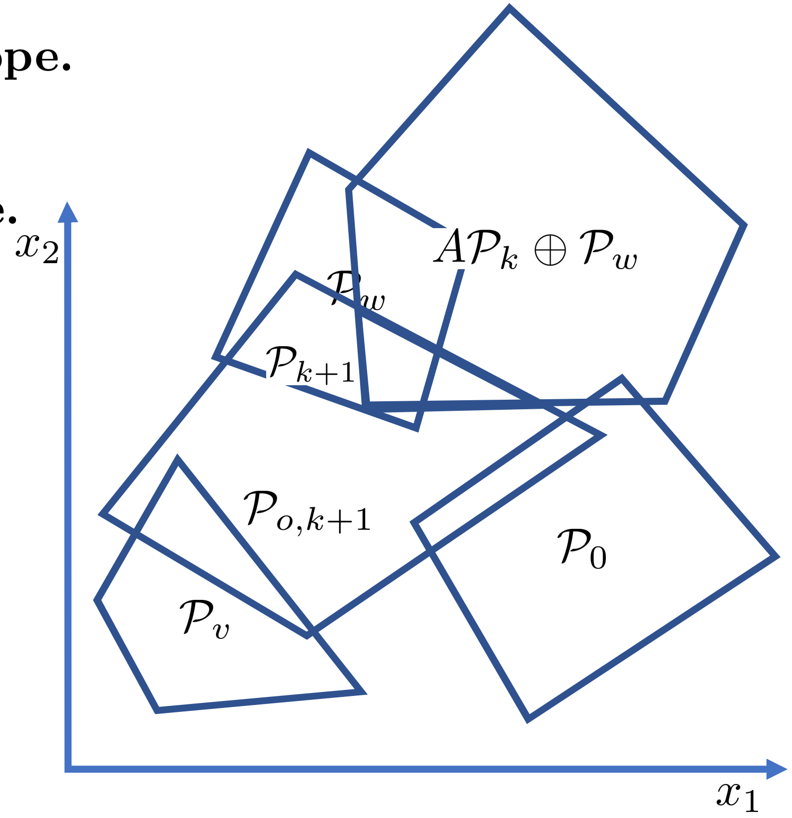
$$\mathcal{P}_{k+1|k}(G_{k+1|k}, h_{k+1|k}) \supseteq A\mathcal{P}_k(G_k, h_k) \oplus \mathcal{P}_\omega(G_\omega, h_\omega)$$

Update step

$$\mathcal{P}_{k+1}\left(G_{k+1}, h_{k+1}\right) \supseteq \mathcal{P}\left(G_{k+1|k}, h_{k+1|k}\right) \cap \mathcal{P}_{o,k+1}(G_o, h_o)$$

Output Polytope

$$\mathcal{P}_{o,k+1}(G_o, h_o) = G_\nu C x_{k+1} \leq h_\nu + G_\nu y_{m,k+1}$$



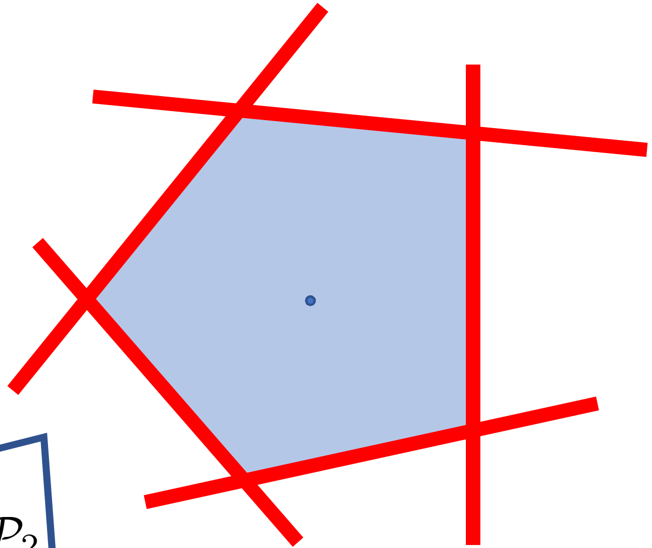
SSE for Nonlinear Systems using Polytopes

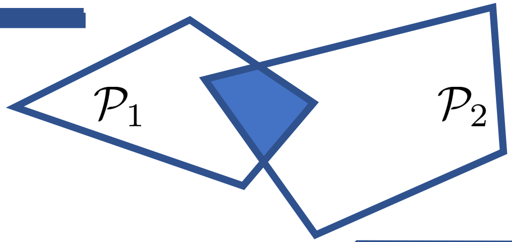
$$(G_{k+1}, h_{k+1}) \Rightarrow \boxed{\underset{G,h,a,b}{\operatorname{argmin}} \sum_{j=1}^{\ell} \|a - b_j\|_2^2}$$

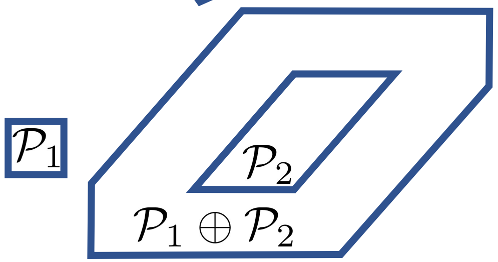
$$\text{s.t.} \begin{cases} \forall j \in \{1, \ldots, \ell\}, \\ G_j b_j = h_j \\ Ga \leq h \\ N_k G_w = M_k G_k A^{-1} \end{cases}$$

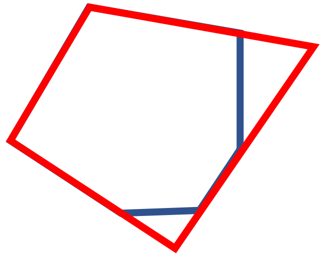
$$G = \Lambda_k \begin{pmatrix} M_k G_k A^{-1} \\ G_v C \end{pmatrix}$$

$$h = \Lambda_k \begin{pmatrix} M_k h_k + N_k h_w \\ h_v + G_v y_{k+1} \end{pmatrix}$$









Case study: A Double Integrator

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ B := \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ C := \begin{pmatrix} 1 & 0 \end{pmatrix},$$

Constraints

$$u \in [-1, 1]$$

$$\omega_k \in \mathcal{P}_w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Initial Conditions

$$x_0 := \begin{pmatrix} 20 \\ 10 \end{pmatrix} \qquad \mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 32 \\ 11 \\ -15 \\ -6 \end{pmatrix}.$$

Output

$$y_k = Cx + \nu_k$$

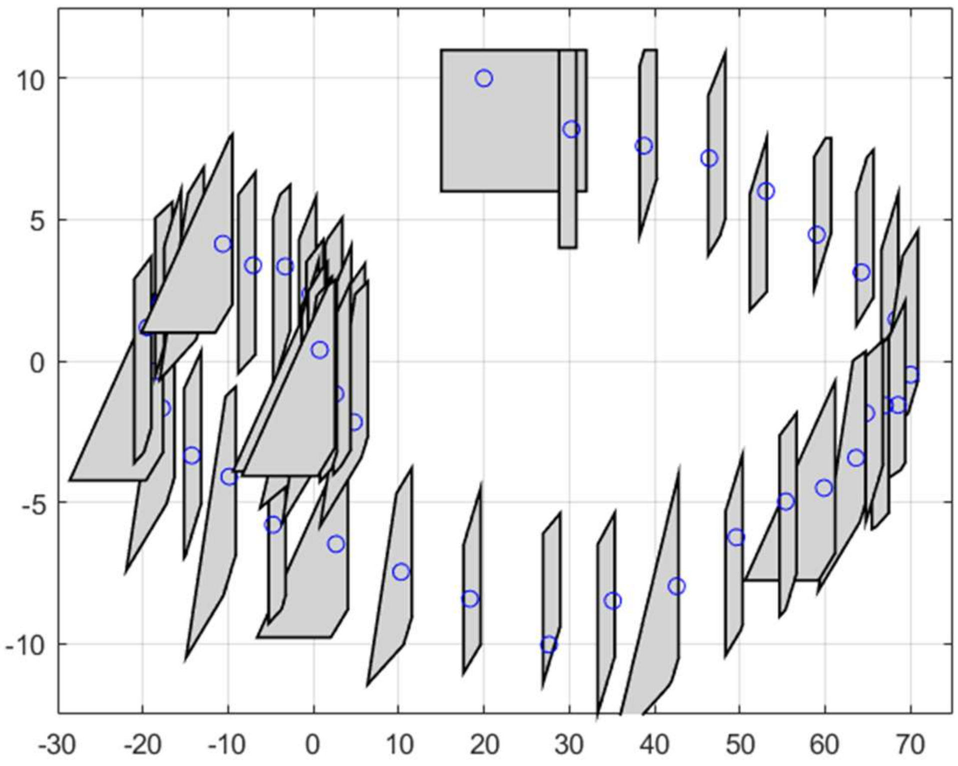
$$\nu_k \in [-1, 1]$$

Control

$Q = I, R = 1$

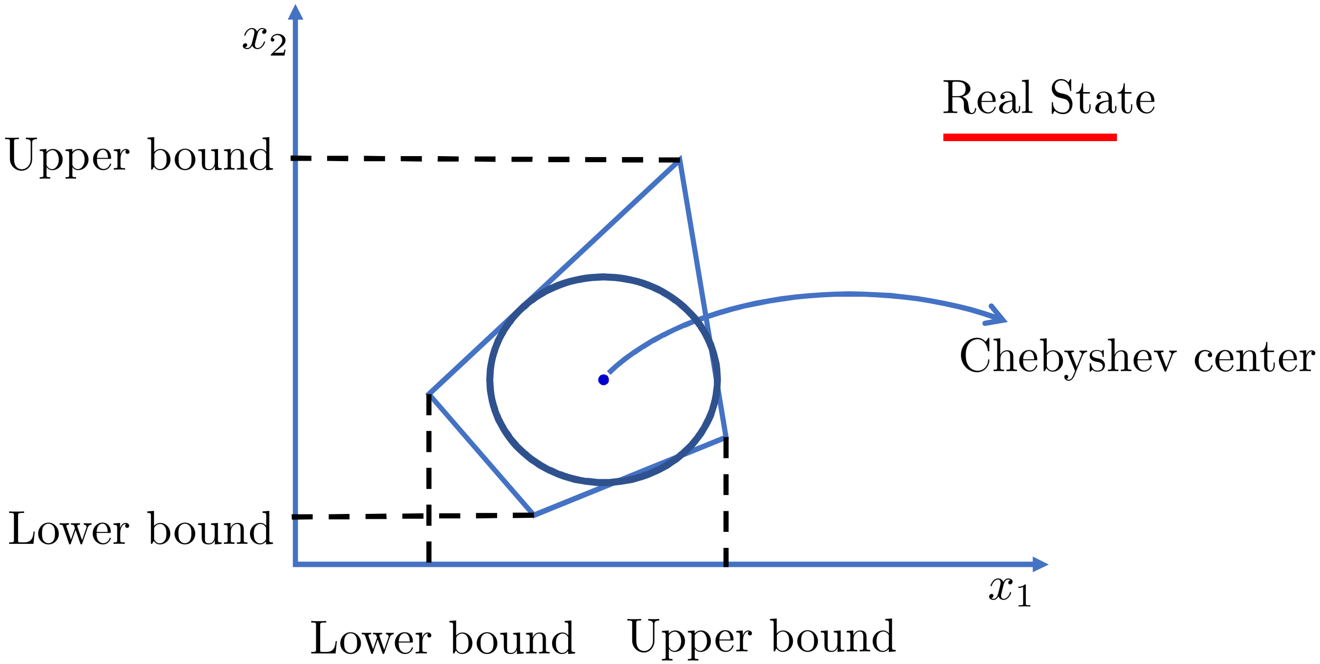
Classic LQR

Results

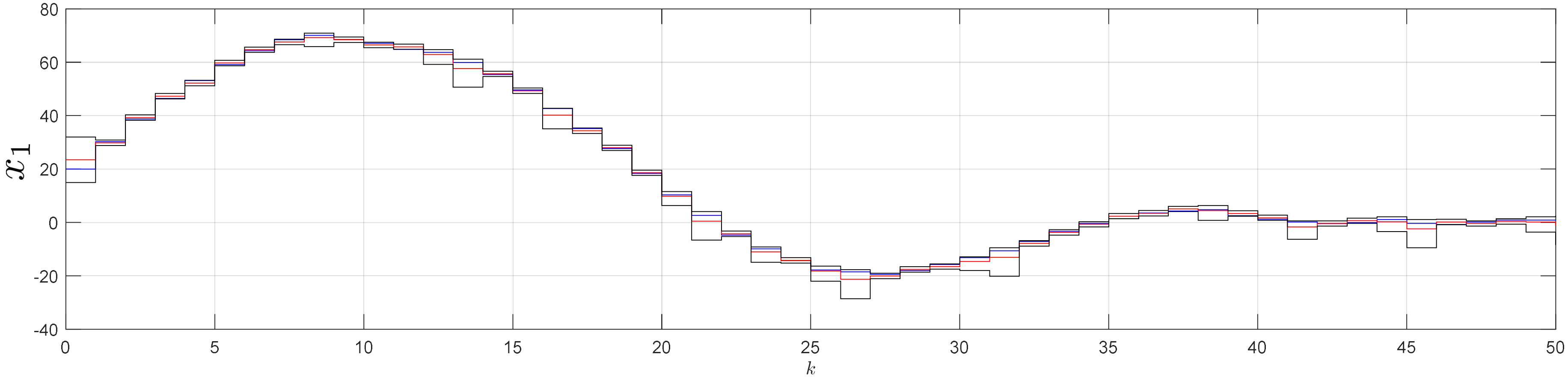


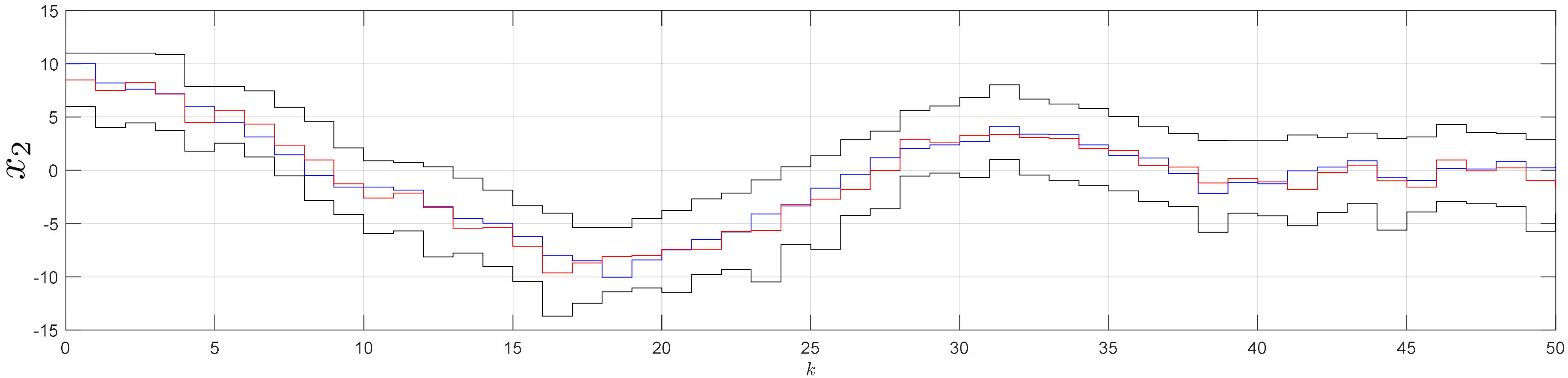
- The method proposed reduced the feasible set.
- In some time steps, the algorithm finds larger approximations than usual. But the result is still consistent and valid.
- In each time step, the corresponding polytope contains the state of the system.
- Our computational experience shows that it is not always possible to find the global solution.

States and Bounds



States and Bounds





A simple comparison

Parallelotopes versus polytopes

$w_k = 0$

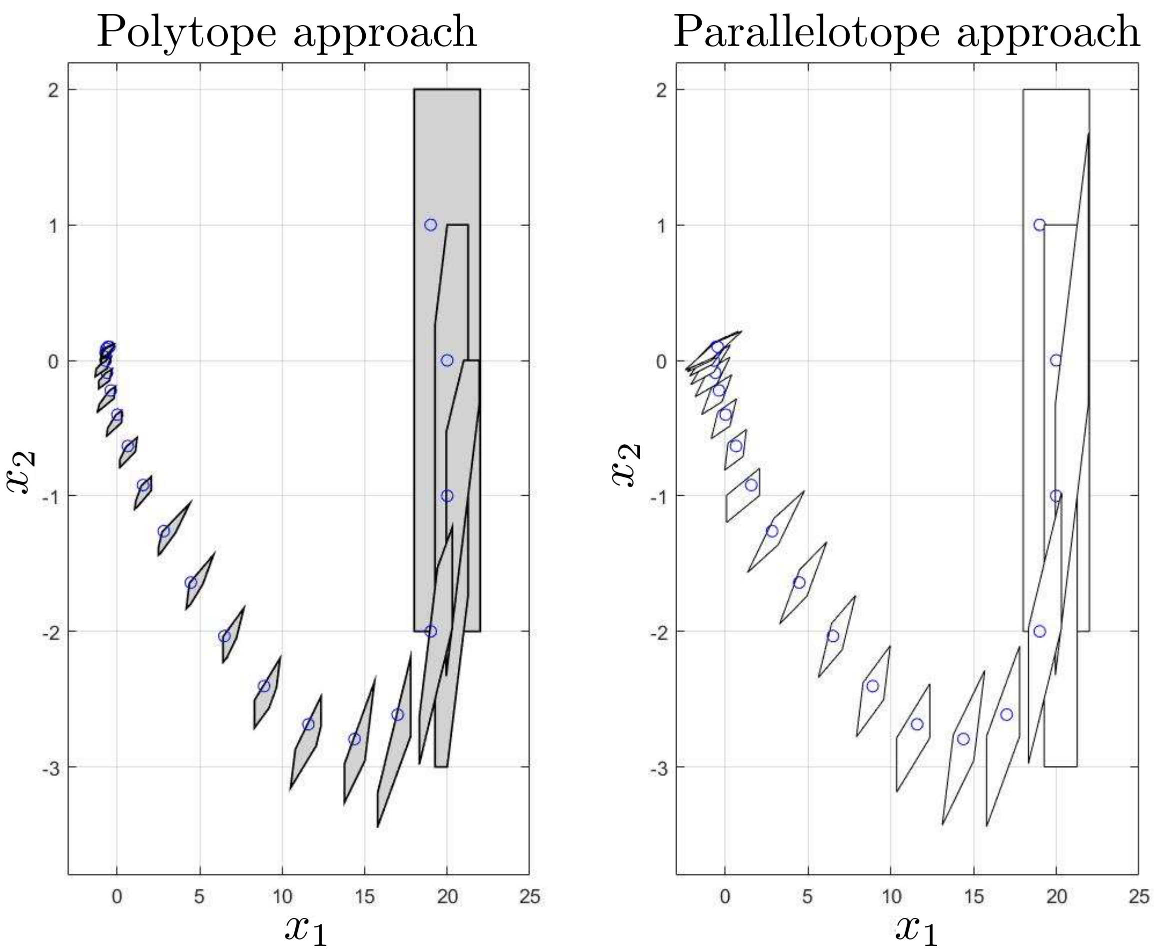
Output

$y_k = Cx + \nu_k$

$\nu_k \in [-1, 1]$

Initial polytope.

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 22 \\ 2 \\ -18 \\ 2 \end{pmatrix}$$



Conclusions

- A new method of set-membership set-based state estimation using polytopes was proposed.
- The estimation method solves only a single NLP to propagate, update, and reduce the polytope in every step time.
- The approach was tested using the double integrator.

Questions:
carlos.valero@stuba.sk

- It was demonstrated that the obtained estimates are consistent and valid.
- The computational complexity of the proposed approach is considerable and good initialization procedures are needed.
- Future works will focus on developing a sophisticated tuning method, as well as, testing the method against other set-membership estimation algorithms.

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